Error Rate Analysis of GF(q) Network Coded Detect-and-Forward Wireless Relay Networks Using Equivalent Relay Channel Models

Ilgın Şafak, *Student Member, IEEE,* Emre Aktaş, *Member, IEEE,* and Ali Özgür Yılmaz, *Member, IEEE*

Abstract

This paper investigates simple means of analyzing the error rate performance of a general q-ary Galois Field network coded detect-and-forward cooperative relay network with known relay error statistics at the destination. Equivalent relay channels are used in obtaining an approximate error rate of the relay network, from which the diversity order is found. Error rate analyses using equivalent relay channel models are shown to be closely matched with simulation results. Using the equivalent relay channels, low complexity receivers are developed whose performances are close to that of the optimal ML receiver.

Index Terms

Cooperative diversity; detection and estimation; source/channel coding.

I. Introduction

Cooperative communication is effective in extending the coverage area and increasing the quality of service of relay networks by means of diversity [1]. The simplest method of trans-

I. Şafak and E. Aktaş are with the Department of Electrical and Electronics Engineering, Hacettepe University, Beytepe Campus, Ankara 06800, Turkey, e-mail: ilgins@hacettepe.edu.tr, eaktas@ee.hacettepe.edu.tr.

A. Ö. Yılmaz is with the Department of Electrical and Electronics Engineering, Middle Eastern Technical University (METU), Ankara 06531, Turkey, e-mail: aoyilmaz@metu.edu.tr.

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mitting a message in a cooperative communication network is routing, where the data received at a node are simply forwarded to another. In network coding (NC), multiple data received at a node can be combined and transmitted simultaneously. Hence, NC can be used to increase the throughput of the system and reduce bandwidth and transmission energy consumption [2].

Non-binary NC is a simple and efficient means of achieving improved diversity gain in wireless relay networks compared to binary network coding. In [3], a q-ary Galois field (GF(q)) NC multi-user cooperative wireless relay network is considered. The users act as each other's relay nodes and use GF(q) NC and decode-and-forward (DF) techniques. The use of non-binary network coding was shown to lead to higher diversity orders compared to binary network coding [3].

Amplify-and-forward (AF) and DF are two commonly used techniques for relaying. In AF relaying, the signal received at the relay is transmitted after being amplified by a factor. In DF, the received signal is decoded then re-encoded and modulated at the relay before re-transmission. AF needs the redesign of many transceivers while DF is computationally complex for the relay node. The detect-and-forward (DetF) technique, where the received symbol is detected and then modulated without decoding and re-encoding, is a low-complexity alternative to DF [1]. In this paper, the DetF method is considered.

Availability of channel state information (CSI) or SNR values at the relay and/or at the destination improves the performance of relay networks significantly. If CSI is available at an AF relay of a source-relay-destination $(S-R-D)$ link, the amplification factor may be chosen to compensate for the instantaneous channel gain of the $S-R$ link [4]. If SNR values of the $S-R$ link are available at the destination in DetF relay networks, the error propagation effects can be mitigated by taking the relay detection error statistics into account at the destination [5]. In [6], the closed-form error probability of a GF(2) NC DetF relay network with known relay error statistics is obtained. The error performance analyses in [5] and [6] are derivable when GF(2) NC and BPSK modulation techniques are employed, however, the analyses are highly complex for higher Galois fields and modulation orders.

In order to analyze the error rate of the relay channel, various analytically tractable approximations may be used. In [7], the max-log approximation is used in the performance analysis of a coded, cooperative, two-hop, DetF relay network with multiple relays. The approximation exhibits accurate error performance for the relay network. However, the approximation in [7] does not consider network-coded relay networks. Another approximation technique is the equivalent

relay channel method, which approximates a two-hop relay channel with a single-hop channel [8]–[13]. The most commonly used equivalent channel model in DF and DetF relay channels is the minimum equivalent relay channel [8]–[10]. In this model, the SNR of the equivalent single-hop channel is equal to the minimum of the $S-R$ and $R-D$ channels' SNR values. The minimum equivalent relay channel model is preferable due to its simplicity. In the equivalent relay channel model in [11], the equivalent SNR value is a function of the error probabilities of the $S-R$ and $R-D$ channels and is shown to asymptotically approach the SNR value of the minimum equivalent relay channel. The equivalent relay channel model is used in [11] to develop a diversity achieving cooperative maximum ratio combining (C-MRC) receiver and in the diversity analysis of C-MRC. However, [11] is short of the diversity order and error rate analysis for ML detection. The equivalent relay channel model developed for multi-hop DF relay networks without network coding in [11], is shown to also be valid for GF(2) NC DF two-way relay channels in [12]. In [13], an equivalent relay channel method called propagation error modeling (PEM) is devised for a low-complexity detector for GF(q) and complex field network coded (CFNC) DF relay networks. In PEM, the detection errors that occur at the relays are modeled as virtual noise at the destination in order to overcome the error propagation effects in DF systems. The PEM method, which displays close performance to the relay network, can be used in obtaining the diversity order of the relay network. However, this method does not readily provide the error rate.

To the best of our knowledge, simple means of obtaining the diversity order and error rate for ML detection of general GF(q) NC DetF relay networks which use relay error statistics at the destination is lacking. Low-complexity alternatives to the optimal receiver for such a relay network are also not available. The aim of this paper is to address these problems.

In this contribution, the error rate analysis methods of [5], [11] are generalized to GF(q) NC DetF relay networks. Two-hop relay channels are approximated using equivalent relay channel models, where the equivalent relay channel model in [11] is adapted for general GF(q) NC DetF relay networks and is compared to the minimum equivalent relay channel model. The equivalent relay models are then used in finding pairwise error probabilities (PEPs), from which the diversity order of ML detection and an approximation to the union bound are obtained. The optimal receiver's error performance is compared with the ML performances of the equivalent relay channel receivers. It is observed that the equivalent channel models successfully approximate

the relay network.

The organization of this paper is as follows. In Section II, the considered system model and data transmission technique are given. In Section III, error rate analyses of relay networks with error-free and error-prone $S-R$ channels are provided; the ML rule is derived and an approximate union bound for error rates is obtained using equivalent relay channel models. In Section IV, the ML rule and union bound symbol error rate (SER) performances of the equivalent relay networks are obtained and compared to the relay network's ML performance. The results are summarized in Section V.

II. GF(q) Network Coded Detect-and-Forward Relay Model

As a general model, the relay network model in Fig. 1 is considered. M-ary phase shift keying (PSK) and time division multiplexing (TDM) techniques are used in the transmission of the GF(q) network coded symbols. All channels are assumed to be independent and Nakagami-m fading. The users act as each other's relay nodes. The DetF method is used in relaying. Imperfect detection at the relays is taken into account, and the detection error statistics observed at the relays are assumed to be available at the destination for ML detection. The users and destination are assumed to have SNR knowledge of the incoming data's channel.

The data of each user terminal T_n, $1 \leq n \leq N$, are transmitted to the destination D, directly and via other users as follows. T_n broadcasts its modulated data to the other terminals and D, where the data received at the terminals are detected. After all users have completed transmitting their data, $K-N$ $(K>N)$ users transmit network coded, modulated data to D using a pre-determined network coding rule, hence a total of K symbols are received at D. The received data at D can be expressed as

$$\mathbf{y_D} = \mathbf{H_D}\,\mathbf{s} + \mathbf{n_D} \tag{1}$$

where $\mathbf{y_D}$ is the $K \times 1$ sized vector consisting of K data received at D:

$$\mathbf{y_D} = [y_{1D,sys} \quad \dots \quad y_{ND,sys} \quad y_{1D,nc} \quad \dots \quad y_{(K-N)D,nc}]^T \tag{2}$$

$y_{nD,sys}$ is the observation at D corresponding to the n^{th} direct transmission, which is the systematic symbol from T_n; $y_{lD,nc}$ is the observation at D corresponding to the l^{th} network coded data, $1 \leq l \leq K-N$; $\mathbf{H_D}$ is the $K \times K$ sized diagonal fading coefficient matrix observed

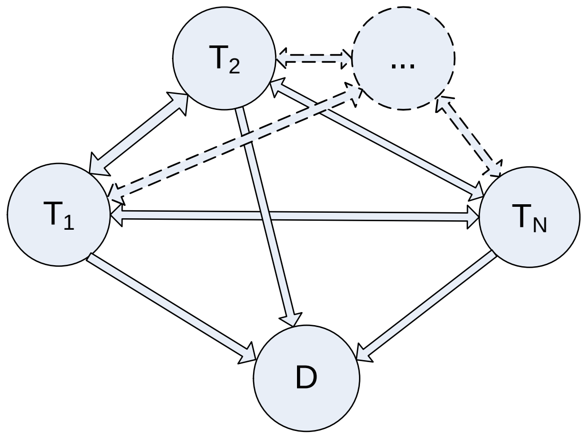

Fig. 1. N-user relay network

at D; $\mathbf{n_D}$ is the $K \times 1$ sized noise vector at D and $\mathbf{s}$ is the $K \times 1$ sized vector comprising the network coded modulated data transmitted from the terminals:

$$\mathbf{s} = \varphi\left(\mathbf{G^T u} + \mathbf{e_R}\right). \tag{3}$$

Here, $\varphi(\cdot)$ denotes the constellation mapping and $\mathbf{G}$ is the generator matrix represents how network coding is performed in the relay network. For the considered network coding operation, $\mathbf{G} = [\mathbf{I} \ \vdots \ \mathbf{P}]_{N \times K}$, where the $N \times N$ identity matrix $\mathbf{I}$ represents the systematic data (first N symbol transmissions), and the $(K-N) \times K$ parity check matrix $\mathbf{P}$ represents the remaining $K-N$ network coded symbols transmitted. The data vector sent by the user terminals is shown by $\mathbf{u} = [u_1 \ \ \ldots \ \ u_N]$, where u_n is the data of T_n; $\mathbf{e_R}$ is the $K \times 1$ sized coded detection error vector at the relays; the elements of $\mathbf{G}, \mathbf{u}$ and $\mathbf{e_R}$ are in GF(q), $1 \leq n \leq N$.

Let us define the set of all modulated GF(q) network coded codewords χ. When the relay network in Fig. 1 has error-free $S-R$ links, there are q^N possible such codewords. Corresponding to each source symbol configuration is a codeword $\mathbf{X}(i)$, where $\mathbf{X}(i)$ is the i^{th} element of the codeword set χ:

$$\mathbf{X}(i) = \varphi(\mathbf{G^T u}) \tag{4}$$

assuming the i^{th} element of the set of all data vectors of $\mathbf{u}$ is transmitted. The aim of the destination is to decide which codeword was sent based on the observation $\mathbf{y_D}$ and the error statistics of the relays, where the decision at D is denoted by $\hat{\mathbf{X}}(j)$.

The relay network in Fig. 1 contains multi-source relay channels due to the use of network coding at the user terminals. For analytical tractability in the error rate analysis, the multi-source relay channels will be approximated by equivalent relay channels as follows.

Data received from the terminals $T_{\tilde{n}}, \tilde{n} \neq n,$ are detected and then GF(q) network coded at the relay terminal $T_n, 1 \leq n, \tilde{n} \leq N$

$$\hat{u}_{nR} = u_{nR} + e_{nR} \tag{5}$$

where u_{nR} is the error-free network coded data at T_n; e_{nR} is the network coded detection error at T_n; $\hat{u}_{nR}, u_{nR}$ and e_{nR} are GF(q) elements. The network coded data received from T_n at D is

$$y_{nD,nc} = h_{nD,nc} s_{nR} + n_{nD,nc} \tag{6}$$

where $h_{nD,nc}$ is the fading coefficient of the $T_n - D$ channel over which the network coded data is sent; $s_{nR} = \varphi(\hat{u}_{nR})$ is the modulated network coded data sent from T_n and $n_{nD,nc}$ is the noise term at the destination of the observed network coded data. When hard-decision ML detection is performed at the destination, the hard-decision of the data in (6) is used

$$y_{nD,nc} \overset{hard-decision}{\rightarrow} z_{nD,nc} \tag{7}$$

where $z_{nD,nc}$ is the hard-decision of $y_{nD,nc}$ at D. The instantaneous end-to-end symbol error probability (SEP) of the relay channel is:

$$\Pr(z_{nD,nc} \neq u_{nR} | u_{nR}) = \sum_{e_{nR}=1}^{q-1} p(z_{nD,nc} \neq u_{nR} | \hat{u}_{nR} = u_{nR} + e_{nR}) p(e_{nR}). \tag{8}$$

The multi-source relay channel described above can be approximated by a single-hop $S - D$ channel as shown in Fig. 2, where T_n corresponds to the relay node R and terminals $T_{\tilde{n}}$ represent the source nodes $S_{\tilde{n}}, \tilde{n} \neq n$. Dropping the $n's$, the instantaneous SEP given in (8) is approximated by the instantaneous SEP of a single-hop channel which has an instantaneous SNR per symbol γ_{eq}:

$$\begin{aligned} P^s_{eq}(\gamma_{eq}) &= \Pr(z_{D,nc} \neq u_R | u_R) \\ &= \sum_{e_R=1}^{q-1} p(z_D \neq u_R | \hat{u}_R = u_R + e_R) p(e_R). \end{aligned} \tag{9}$$

Two equivalent relay channel models will be used in the error rate analysis, which will be discussed in Section III.

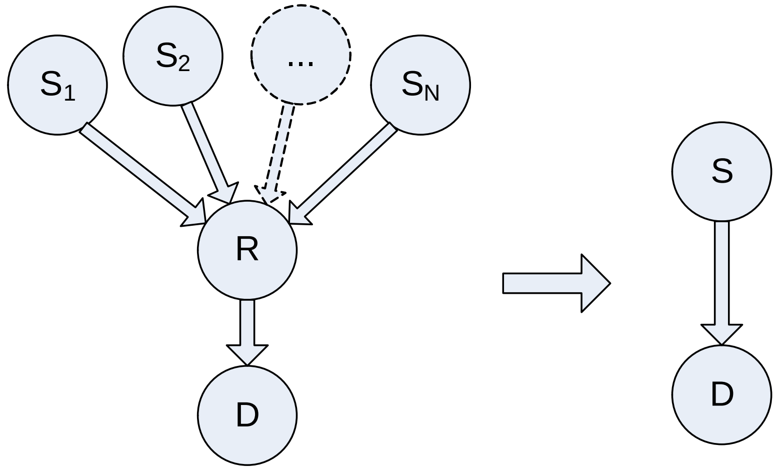

Fig. 2. Equivalent relay channel of a relay channel with N sources

III. Error Rate Analysis

Before investigating the error rate analysis, we provide the soft- and hard-decision ML rules for relay networks with error-free $S-R$ links, which will later on be used in deriving the ML rules for relay networks with error-prone $S-R$ links.

Soft-decision ML is defined as ML detection at the destination based on the soft-decision observation vector $\mathbf{y_D}$ in (1). The soft-decision ML rule of terminal T_i's data u_i, $1 \leq i \leq N$, is given as:

$$u_i^{ML} = \arg\max_{u_i} p(\mathbf{y_D}|u_i). \tag{10}$$

Using the independence properties, the likelihood function given above is expanded

$$\begin{aligned} p(\mathbf{y_D}|u_i) &= \sum_{\mathbf{u^i}} p(\mathbf{y_D}|u_i, \mathbf{u^i}) p(\mathbf{u^i}) \\ &= \sum_{\mathbf{u^i}} \prod_{i=1}^{K} p(y_{D,i}|u_i, \mathbf{u^i}) p(\mathbf{u^i}) \end{aligned} \tag{11}$$

where $\mathbf{u^i} = \mathbf{u} \setminus u_i$ is the data vector excluding u_i and $y_{D,i}$ is the i^{th} element of $\mathbf{y_D}$. Assuming q-ary symbols of all N users are transmitted equally likely and independently, the joint distribution of $\mathbf{u^i}$ is:

$$p(\mathbf{u^i}) = \left(\frac{1}{q}\right)^{N-1}. \tag{12}$$

Inserting (12) into (11) gives:

$$p(\mathbf{y_D}|u_i) = q^{-(N-1)} \sum_{\mathbf{u^i}} \prod_{i=1}^{K} p(y_{D,i}|u_i, \mathbf{u^i}) \tag{13}$$

The hard-decision ML rule is defined as ML detection at the destination, based on the hard-decision observation vector $\mathbf{z_D}$, where each element is the individual hard-estimate of the corresponding element of $\mathbf{y_D}$:

$$\mathbf{y_D} \overset{hard-decision}{\rightarrow} \mathbf{z_D}. \tag{14}$$

The hard-decision ML rule of terminal T_i's data u_i, $1 \leq i \leq N$ is

$$u_i^{ML} = \arg\max_{u_i} p(\mathbf{z_D}|u_i) \tag{15}$$

where the likelihood function given in (15) is expanded using the independence properties

$$p(\mathbf{z_D}|u_i) = \sum_{\mathbf{u^i}} \prod_{i=1}^{K} p(z_{D,i}|u_i, \mathbf{u^i}) p(\mathbf{u^i}). \tag{16}$$

A. Relay Networks with Error-Free Source-Relay Channels

The error rate analysis of the relay network with error-free $S-R$ links is provided as a preliminary to the error rate analysis of the relay network with error-prone $S-R$ links.

In the error rate analysis of the relay network with error-free $S-R$ links, the reliability of the observation at the destination depends only on the $R-D$ channel SNR values. The transmitted block in Euclidean space is the modulated codeword in (4). The symbol error probability (SEP), is upper bounded by the average of the average pairwise error probabilities (PEPs) over a single symbol error event [14]

$$P_S \leq \frac{1}{K\,|\chi|} \sum_{k=1}^{K} \sum_{\mathbf{X}(i),\hat{\mathbf{X}}(j) \epsilon \chi, j_k \neq i_k} P\left(\mathbf{X}(i) \rightarrow \hat{\mathbf{X}}(j)\right) \tag{17}$$

where $\mathbf{X}(i) = \varphi(i_1 \ldots i_K)$ is the transmitted modulated codeword and $\hat{\mathbf{X}}(j) = \varphi(j_1 \ldots j_K)$ is the decision at D. The codewords $\mathbf{X}(i)$ and $\hat{\mathbf{X}}(j)$ are found using (4) and i_k and j_k are GF(q) elements, $1 \leq k \leq K$. Note that each PEP $P\left(\mathbf{X}(i) \rightarrow \hat{\mathbf{X}}(j)\right)$ in (17) is obtained after averaging over the channel $\mathbf{H_D}$. This averaging can be performed using the moment generating function (MGF) method [14], [15]:

$$P\left(\mathbf{X}(i) \rightarrow \hat{\mathbf{X}}(j)\right) = \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \prod_{k=1}^{K} M_k \left(-\frac{|\varphi(i_k) - \varphi(j_k)|^2}{4\sin^2(\theta)}\right) \mathrm{d}\theta. \tag{18}$$

Assuming a Nakagami-m fading environment, $M_k(s) = (1 - s\bar{\gamma}_k/m)^{-m}$ and $\bar{\gamma}_k$ are the MGF of the instantaneous SNR and average SNR value of the k^{th} element of the received data vector in (1), respectively and m is the fading figure.

Obtaining the PEPs is not as straightforward, however, when the $S-R$ channels are error-prone. The equivalent relay models described in the next section can be used to overcome this difficulty.

B. Relay Networks with Error-Prone Source-Relay Channels

In relay networks with error-prone $S-R$ channels, when no relay error statistics are available at the destination, the soft-decision ML rule is equivalent to that used in the error-free case, given by (11). However, if the relay error statistics are known at D, the likelihood function for the soft-decision ML rule is

$$\begin{aligned} p(\mathbf{y_D}|u_i) &= \sum_{\mathbf{e_R}} p(\mathbf{y_D}|u_i, \mathbf{e_R})p(\mathbf{e_R}) \\ &= \sum_{\mathbf{e_R}} \sum_{\mathbf{u^i}} p(\mathbf{y_D}|u_i, \mathbf{u^i}, \mathbf{e_R})p(\mathbf{u^i})p(\mathbf{e_R}) \end{aligned} \tag{19}$$

where the fact that relay events are independent from the source symbols is used. Using (12):

$$p(\mathbf{y_D}|u_i) = q^{-(N-1)} \sum_{\mathbf{e_R}} \sum_{\mathbf{u^i}} p(\mathbf{y_D}|u_i, \mathbf{u^i}, \mathbf{e_R})p(\mathbf{e_R}). \tag{20}$$

Similarly, for the hard-decision ML rule, when the relay error statistics are available at D, the likelihood function is:

$$p(\mathbf{z_D}|u_i) = q^{-(N-1)} \sum_{\mathbf{e_R}} \sum_{\mathbf{u^i}} p(\mathbf{z_D}|u_i, \mathbf{u^i}, \mathbf{e_R})p(\mathbf{e_R}). \tag{21}$$

Compared to (13) and (16), the likelihood functions in (20) and (21) involve the averaging over $\mathbf{p}(\mathbf{e_R})$, which increases the mathematical complexity of the soft- and hard-ML rules, respectively, hence increasing the computational complexity of the optimal soft- and hard-receivers. The computational complexity of the receivers can be decreased using equivalent relay channel models, which will be discussed later on.

For the relay networks with error-prone $S-R$ channels, the error rate analysis is not as simple as in the error-free case. This is due to two facts. First, the optimal ML receiver is complex because of the averaging of $S-R$ link errors. Second, even if a suboptimal receiver is used which does not perform this averaging, the analysis is still complex due to the $S-R$ error

events. In order to obtain a tractable analytical results for the PEPs, we propose using equivalent relay channel models both for the ML receiver and for its analysis. Thus, we will approximate the error-prone relay channel with an equivalent error-free relay channel model, and analyze the performance of the optimal receiver receiver for that model.

In [11], an equivalent relay channel model is proposed that approximates a multi-hop, cooperative DF relay channel with a single-hop channel. The model is shown to be valid for any modulation order, assuming coherent modulation/demodulation. The equivalent relay channel takes the channel model of the $S-R$ and $R-D$ channels, which is exemplified for a flat Rayleigh fading, two-hop relay channel with coherent BPSK modulation/demodulation, as shown in Fig. 3. The instantaneous bit-error probability (BEP) of the equivalent relay channel is found as [11]

$$P^b_{Q^{-1}} = P^b_{SR}\left(1-P^b_{RD}\right) + P^b_{RD}\left(1-P^b_{SR}\right) \tag{22}$$

where P^b_{SR} and P^b_{RD} denote the BEP's of the $S-R$ and $R-D$ links, respectively. Assuming coherent BPSK modulation/demodulation, the equivalent BEP expressed above is approximated to a single-hop channel that has an instantaneous SNR per bit $\gamma^b_{Q^{-1}}$ [11]:

$$\begin{aligned} P^b_{Q^{-1}} &= Q\left(\sqrt{2\gamma^b_{SR}}\right)\left(1-Q\left(\sqrt{2\gamma^b_{RD}}\right)\right) + Q\left(\sqrt{2\gamma^b_{RD}}\right)\left(1-Q\left(\sqrt{2\gamma^b_{SR}}\right)\right) \\ &= Q\left(\sqrt{2\gamma^b_{Q^{-1}}}\right) \end{aligned} \tag{23}$$

where γ^b_{SR} and γ^b_{RD} denote the instantaneous SNR's per bit of the $S-R$ and $R-D$ links, respectively. From (23), an instantaneous SNR value of the equivalent relay channel is found as [11]

$$\gamma_{Q^{-1}} = \frac{1}{2}\{Q^{-1}(P^b_{Q^{-1}})\}^2. \tag{24}$$

Due to the use of the Gaussian Q-inverse function in (24), the equivalent relay channel model is referred to as the Q-inverse equivalent relay channel in the rest of this paper. Notice from (22) and (23) that, for a fading channel, the instantaneous SNRs γ^b_{SR} and γ^b_{RD} are random variables, thus the instantaneous SNR $\gamma^b_{Q^{-1}}$ is also a random variable. For the error rate analysis, the distribution of $\gamma^b_{Q^{-1}}$ is approximated by a conventional fading SNR distribution whose parameters are determined.

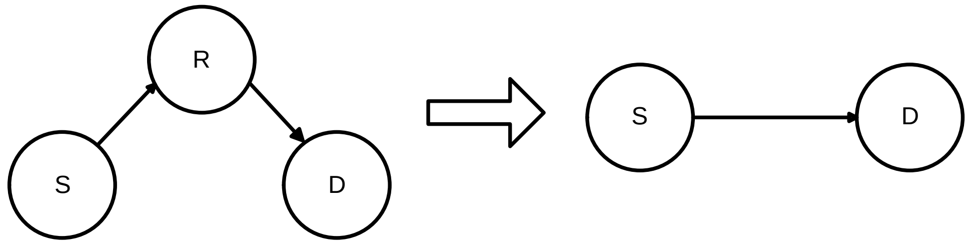

Fig. 3. Equivalent relay channel of a single-sourced relay channel

The Q-inverse equivalent relay channel model, originally developed for DF relay networks, is used in a GF(2) NC DF two-way relay channel in [12] and has the same BEP expression as in (23) and SNR value in (24).

The novel part of this paper involves the adaptation of the Q-inverse equivalent relay channel model to GF(q) NC, coherent M-PSK modulated, DetF, Nakagami-m fading relay channels with multiple sources. The proposed equivalent relay channel model is used in obtaining approximate error rate expressions for GF(q) NC, coherent M-PSK modulated, DetF, Nakagami-m fading relay networks. Soft- and hard-decision ML receivers employing the Q-inverse equivalent relay channel model, which display very close performance to the optimal receiver, are developed. The instantaneous SEP in (8) is approximated to the instantaneous SEP of a single channel

$$\begin{aligned} P^s_{Q^{-1}} &= 2\,Q\left(\sqrt{2\gamma_{Q^{-1}}}\,\sin\left(\pi/M\right)\right) \\ &= \Pr(z_{D,nc} \neq u_R | u_R) \end{aligned} \tag{25}$$

for coherent M-ary PSK for large values of M and SNR [1]. The instantaneous SNR per symbol of the Q-inverse equivalent relay channel is found using (25) as:

$$\begin{aligned} \gamma_{Q^{-1}} &= \frac{1}{2\sin^2\left(\pi/M\right)}\left\{Q^{-1}(0.5P^s_{Q^{-1}})\right\}^2 \\ &= \frac{1}{2\sin^2\left(\pi/M\right)}\left\{Q^{-1}\left(0.5\sum_{e_R=1}^{q-1} p(z_{D,nc} \neq u_R | \hat{u}_R = u_R + e_R)p(e_R)\right)\right\}^2 \end{aligned} \tag{26}$$

The average value of (26)

$$\bar{\gamma}_{Q^{-1}} = \frac{1}{2\sin^2\left(\pi/M\right)} E\left[\left\{Q^{-1}(0.5P^s_{Q^{-1}})\right\}^2\right] \tag{27}$$

has no known closed-form expression but can easily be obtained numerically offline and used in online calculations with a look-up table.

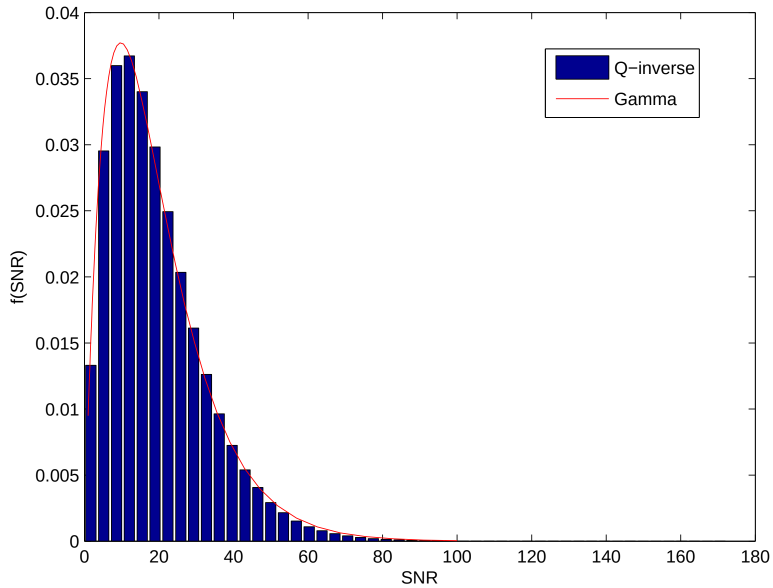

Fig. 4. Histogram of the Q-inverse equivalent channel's instantaneous SNR value in a Nakagami-m fading environment (m=2)

In a Nakagami-m fading environment, it is seen in the histogram result in Fig. 4 that the distribution of $\gamma_{Q^{-1}}$ is well approximated by a Gamma distribution when $m = 2$, so the equivalent relay channel can be approximated as a single-hop Nakagami-m channel. The histogram is obtained via Monte Carlo simulation of the instantaneous SNR in (26). Based on this result, the instantaneous SNR value in (26) is assumed to be Gamma distributed:

$$f_{\gamma_{Q^{-1}}}(\gamma) = \frac{\gamma^{m-1}}{\Gamma(m)} \left(\frac{m}{\bar{\gamma}_{Q^{-1}}}\right)^m \exp\left(-\frac{m\gamma}{\bar{\gamma}_{Q^{-1}}}\right), \quad \gamma \geq 0. \tag{28}$$

The Q-inverse equivalent hard-receiver of the relay network in Fig. 1 obtains the ML estimates of u_i with the hard-decision ML rule in (15), where the SEP terms of network coded data in the likelihood function in (21) are approximated by (25). The Q-inverse equivalent hard-receiver is illustrated with an example as follows. Consider the QPSK modulated, 2-user GF(4) NC DetF relay network in Fig. 1 for $N = 2$, which has a generator matrix:

$$\mathbf{G_1} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}. \tag{29}$$

Hence, network coding is performed at T_1 and T_2 as $u_{1R} = u_1 + 2u_2$ and $u_{2R} = u_1 + u_2$,

respectively, where $u_1, u_2, u_{1R}, u_{2R} \in \mathrm{GF}(4)$. The received data vector at D is detected as

$$[y_{1D,sys} \quad y_{2D,sys} \quad y_{1D,nc} \quad y_{2D,nc}]^T \overset{hard-decision}{\longrightarrow} [z_{1D,sys} \quad z_{2D,sys} \quad z_{1D,nc} \quad z_{2D,nc}]^T \tag{30}$$

where $y_{nD,sys}$ and $y_{nD,nc}$ denote the systematic and network coded data received from T_n at D, respectively; $z_{nD,sys}$ and $z_{nD,nc}$ are the hard-decision data obtained by individually detecting $y_{nD,sys}$ and $y_{nD,nc}$ respectively, $n = 1, 2$. The likelihood functions of the detected data in (30) are used in the hard-decision ML rule in (15), where the likelihood functions of the network coded data are approximated using the Q-inverse equivalent channel method as follows. The instantaneous SEP of the relay channel $T_2 - T_1 - D$, over which u_{1R} is sent, is approximated using (25):

$$\begin{aligned} P^s_{Q^{-1},1} &= 2Q\left(\sqrt{\gamma_{Q^{-1},1}}\right) \\ &= \sum_{e_{1R}=1}^{3} p\left(z_{1D,nc} \neq u_{1R} | \hat{u}_{1R} = u_{1R} + e_{1R}\right) p(e_{1R}) \end{aligned} \tag{31}$$

where e_{1R} is the network coded coherent QPSK demodulation error at T_1 and $\gamma_{Q^{-1},1}$ is the instantaneous SNR per symbol of the equivalent relay channel of $T_2 - T_1 - D$, which is found as

$$\begin{aligned} \gamma_{Q^{-1},1} &= \{Q^{-1}(0.5\, P^s_{Q^{-1},1})\}^2 \\ &= \left\{Q^{-1}\left(0.5 \sum_{e_{1R}=1}^{3} p\left(z_{1D,nc} \neq u_{1R} | \hat{u}_{1R} = u_{1R} + e_{1R}\right) p(e_{1R})\right)\right\}^2. \end{aligned} \tag{32}$$

Similarly, the instantaneous SEP of the relay channel $T_1 - T_2 - D$, over which u_{2R} is sent, is approximated

$$\begin{aligned} P^s_{Q^{-1},2} &= 2Q\left(\sqrt{\gamma_{Q^{-1},2}}\right) \\ &= \sum_{e_{2R}=1}^{3} p\left(z_{2D,nc} \neq u_{2R} | \hat{u}_{2R} = u_{2R} + e_{2R}\right) p(e_{2R}) \end{aligned} \tag{33}$$

where e_{2R} is the network coded coherent QPSK demodulation error at T_2 and $\gamma_{Q^{-1},2}$ is the instantaneous SNR per symbol of the equivalent relay channel of $T_1 - T_2 - D$:

$$\begin{aligned} \gamma_{Q^{-1},2} &= \{Q^{-1}(0.5\, P^s_{Q^{-1},2})\}^2 \\ &= \left\{Q^{-1}\left(0.5 \sum_{e_{2R}=1}^{3} p\left(z_{2D,nc} \neq u_{2R} | \hat{u}_{2R} = u_{2R} + e_{2R}\right) p(e_{2R})\right)\right\}^2. \end{aligned} \tag{34}$$

Another equivalent relay channel model is the minimum equivalent relay channel model, which is commonly used in analyzing the error performance of DetF relay networks [8]–[10].

The minimum equivalent relay channel has a lower computational complexity than the Q-inverse equivalent relay channel, which is preferable in the receiver design. In this model, the two-hop relay channel shown in Fig. 3 is approximated to a single-hop channel, where the equivalent instantaneous SNR per symbol, γ_{min}, is chosen as the minimum of the $S-R$ and $R-D$ channels instantaneous SNR values [8]–[10]

$$\gamma_{min} = \min\{\gamma_{SR}, \gamma_{RD}\}. \tag{35}$$

Here, γ_{SR} and γ_{RD} are the instantaneous SNR's per symbol of the $S-R$ and $R-D$ channels, respectively. In a Nakagami-m fading environment, the SNR value in (35) is distributed as [16]

$$\begin{aligned} f_{\gamma_{min}}(\gamma) = &\frac{1}{\Gamma(m_{SR})\,\gamma}\left(\frac{m_{SR}\,\gamma}{\bar{\gamma}_{SR}}\right)^{m_{SR}} e^{-\left(\frac{m_{SR}}{\bar{\gamma}_{SR}}+\frac{m_{RD}}{\bar{\gamma}_{RD}}\right)\gamma} \sum_{k=0}^{m_{RD}-1}\frac{1}{k!}\left(\frac{m_{RD}\gamma}{\bar{\gamma}_{RD}}\right)^{k} \\ &+ \frac{1}{\Gamma(m_{RD})\,\gamma}\left(\frac{m_{RD}\,\gamma}{\bar{\gamma}_{RD}}\right)^{m_{RD}} e^{-\left(\frac{m_{SR}}{\bar{\gamma}_{SR}}+\frac{m_{RD}}{\bar{\gamma}_{RD}}\right)\gamma} \sum_{k=0}^{m_{SR}-1}\frac{1}{k!}\left(\frac{m_{SR}\gamma}{\bar{\gamma}_{SR}}\right)^{k} \end{aligned} \tag{36}$$

where m_{SR} and m_{RD} are the fading figures of the $S-R$ and $R-D$ links respectively; $\bar{\gamma}_{SR}$ and $\bar{\gamma}_{RD}$ are the average SNR values of the $S-R$ and $R-D$ links, respectively. When $m_{SR} = m_{RD} = m$, the distribution in (36) is reduced to

$$\begin{aligned} f_{\gamma_{min}}(\gamma) = &\frac{(m\gamma)^m}{\Gamma(m)\gamma} e^{-\left(\frac{1}{\bar{\gamma}_{SR}}+\frac{1}{\bar{\gamma}_{RD}}\right)m\gamma} \\ &\times \left\{\frac{1}{(\bar{\gamma}_{SR})^m}\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{m\gamma}{\bar{\gamma}_{RD}}\right)^k + \frac{1}{(\bar{\gamma}_{RD})^m}\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{m\gamma}{\bar{\gamma}_{SR}}\right)^k\right\}. \end{aligned} \tag{37}$$

The average value of (35) when $m_{SR} = m_{RD} = m$

$$\begin{aligned} \bar{\gamma}_{min} = &-2(-1)^{-m}\frac{\Gamma(2m)}{\Gamma(m)^2}\left\{\bar{\gamma}_{RD}\,\mathrm{B}\left(-\frac{\bar{\gamma}_{SR}}{\bar{\gamma}_{RD}}, m+1, -2m\right)\right. \\ &\left. + \bar{\gamma}_{SR}\,\mathrm{B}\left(-\frac{\bar{\gamma}_{RD}}{\bar{\gamma}_{SR}}, m+1, -2m\right)\right\} \end{aligned} \tag{38}$$

where $\mathrm{B}(x, y, z)$ is the incomplete Beta function.

The minimum equivalent relay of the N-source relay channel in Fig. 2 has the instantaneous SNR

$$\gamma_{min} = \min\{\gamma_{S_1R}, \gamma_{S_2R}, \ldots \gamma_{S_NR}, \gamma_{RD}\} \tag{39}$$

where γ_{S_nR} is the instantaneous SNR per symbol of the channel $S_n - R$, $1 \leq n \leq N$. The distribution of (39) and its average SNR value, however, is not as readily found for large values of

N, making the model unsuitable for error rate analysis but useful for designing a low-complexity receiver, which is described below.

The minimum equivalent hard-receiver of the relay network in Fig. 1 obtains the ML estimates of u_i by the hard-decision ML rule in (15), where the SEP terms of the network coded data in the likelihood function in (21) are approximated to the coherent M-PSK demodulation SEP of a minimum equivalent relay channel which has an instantaneous SNR value per symbol γ_{min}:

$$\begin{aligned} P^s_{min} &= Q(\sqrt{2\gamma_{min}}) + \frac{2}{\pi}\int_0^\infty \exp\left[-\left(u-\sqrt{\gamma_{min}}\right)^2\right] Q\left(\sqrt{2}u\tan(\pi/M)\right)\mathrm{d}u \\ &= \sum_{e_R=1}^{q-1} p\left(z_{D,nc}|\hat{u}_R = u_R + e_R\right) p(e_R). \end{aligned} \tag{40}$$

An approximate error rate of the relay network in Fig. 1 is found using an equivalent relay channel model in the following manner. The channel over which the network coded data at the terminals are sent to the destination are approximated using an equivalent channel model. The average PEPs of the union bound in (17) are approximated using the same method employed in the error-free case, given by (18), except that the SNR vector is obtained using the equivalent relay channels.

Let us illustrate how the Q-inverse and minimum equivalent relay channels are used in obtaining the PEPs with an example. Consider the 2-user GF(4) NC, DetF, QPSK modulated, relay network described in previously in (29)-(34) assuming a Nakagami-m fading environment with $m = 1$. The relay network has a generator matrix given by (29) and the observed SNR vector at D is

$$\mathbf{\Gamma} = [\gamma_{1D,sys} \quad \gamma_{2D,sys} \quad \gamma_{eq,1} \quad \gamma_{eq,2}] \tag{41}$$

where $\gamma_{nD,sys}$ is the instantaneous SNR value per symbol of the channel over which u_n is transmitted and $\gamma_{eq,n}$ is the instantaneous SNR value per symbol of the equivalent relay channel over which u_{nR} is sent, $n = 1, 2$. Assuming the $S-R$ and $R-D$ channels have equal average SNR values, the average values of (41) are $\bar{\gamma}_{1D,sys} = \bar{\gamma}_{2D,sys} = \bar{\gamma}$ and $\bar{\gamma}_{eq,1} = \bar{\gamma}_{eq,2} = \bar{\gamma}_{eq}$ and the corresponding average SNR vector is

$$\bar{\mathbf{\Gamma}} = [\bar{\gamma} \quad \bar{\gamma} \quad \bar{\gamma}_{eq} \quad \bar{\gamma}_{eq}]. \tag{42}$$

Inserting $m = 1$ and $\bar{\gamma}_{SR} = \bar{\gamma}_{RD} = \bar{\gamma}$ in (38), the minimum equivalent relay channel's average SNR value is found as $\bar{\gamma}_{min} = \bar{\gamma}/2$, hence the average SNR vector when the minimum equivalent

relay channel model is used is the following:

$$\bar{\mathbf{\Gamma}}_{\mathbf{min}} = [\bar{\gamma} \quad \bar{\gamma} \quad \bar{\gamma}/2 \quad \bar{\gamma}/2]. \tag{43}$$

The average SNR vector when the Q-inverse equivalent relay channel model is used is given below:

$$\bar{\mathbf{\Gamma}}_{\mathbf{Q^{-1}}} = [\bar{\gamma} \quad \bar{\gamma} \quad \bar{\gamma}_{Q^{-1}} \quad \bar{\gamma}_{Q^{-1}}] \tag{44}$$

where $\bar{\gamma}_{Q^{-1}}$ is found by inserting (32) and (34) into (27) for $M = 4$. The average PEP between the codewords $\mathbf{X}(0) = \varphi([0 \quad 0 \quad 0 \quad 0])$ and $\hat{\mathbf{X}}(1) = \varphi([0 \quad 1 \quad 2 \quad 1])$ is found using the MGF method in (18):

$$\begin{aligned} P\left(\mathbf{X}(0) \to \hat{\mathbf{X}}(1)\right) &= \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \left(1 + \bar{\gamma}\frac{|\varphi(0) - \varphi(0)|^2}{4\sin^2(\theta)}\right)^{-1} \left(1 + \bar{\gamma}\frac{|\varphi(0) - \varphi(1)|^2}{4\sin^2(\theta)}\right)^{-1} \\ &\quad \times \left(1 + \bar{\gamma}_{eq}\frac{|\varphi(0) - \varphi(2)|^2}{4\sin^2(\theta)}\right)^{-2} \mathrm{d}\theta \\ &= \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \left(\frac{2\sin^2(\theta)}{2\sin^2(\theta) + \bar{\gamma}}\right) \left(\frac{2\sin^2(\theta)}{2\sin^2(\theta) + \bar{\gamma}_{eq}}\right)^2 \mathrm{d}\theta \end{aligned} \tag{45}$$

where the modulated codewords are taken as $\varphi(0) = 1$, $\varphi(1) = j$, $\varphi(2) = -j$ and $\varphi(3) = -1$, assuming Gray coding. When the minimum equivalent relay SNR vector in (43) is used in (45), the PEP reduces to

$$P\left(\mathbf{X}(0) \to \hat{\mathbf{X}}(1)\right) = \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \left(\frac{2\sin^2(\theta)}{2\sin^2(\theta) + \bar{\gamma}}\right) \left(\frac{4\sin^2(\theta)}{4\sin^2(\theta) + \bar{\gamma}}\right)^2 \mathrm{d}\theta. \tag{46}$$

Inserting the Q-inverse equivalent SNR vector in (44) into (45) gives

$$P\left(\mathbf{X}(0) \to \hat{\mathbf{X}}(1)\right) = \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \left(\frac{2\sin^2(\theta)}{2\sin^2(\theta) + \bar{\gamma}}\right) \left(\frac{2\sin^2(\theta)}{2\sin^2(\theta) + \bar{\gamma}_{Q^{-1}}}\right)^2 \mathrm{d}\theta. \tag{47}$$

For high SNR values, using the fact that the Q-inverse equivalent relay channel converges to the minimum equivalent relay channel [11], the union bound from (46) and (47) is found using Formula 3.621.3 on p.389 in [17]:

$$P\left(\mathbf{X}(0) \to \hat{\mathbf{X}}(1)\right) \leq \frac{32}{\pi\bar{\gamma}^3} \int_{\theta=0}^{\pi/2} \sin^6(\theta)\mathrm{d}\theta = \frac{32}{\pi\bar{\gamma}^3}\frac{\pi\, 5!!}{2\, 6!!} = \frac{5}{\bar{\gamma}^3}. \tag{48}$$

It is seen after this analysis that the diversity order in this case is equal to 3, from (48).

IV. Performance Results

In this section, optimal receiver's error performance of the relay network shown in Fig. 1 is compared to the error performances of the minimum and Q-inverse receivers and the approximate union bounds obtained using the equivalent relay channel models. For each SNR value, at least 50 bit/symbol errors were observed in obtaining the corresponding BER/SER.

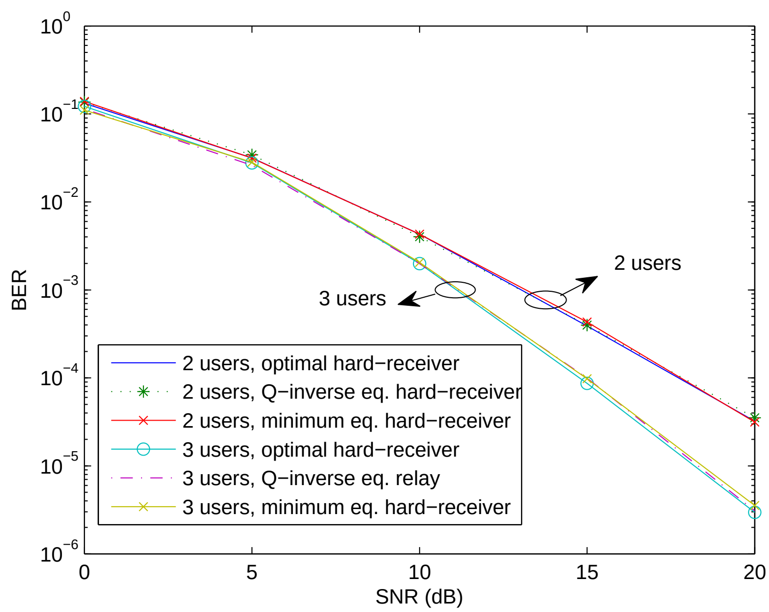

Fig. 5. 2 and 3-user GF(2) NC DetF Nakagami-m (m=1) fading relay networks' hard-decision ML performances

Fig. 5 compares the optimum hard-receivers' BER performances of the 2- and 3-user cases of the GF(2) NC DetF Nakagami-m fading with $m = 1$ (Rayleigh flat-fading) relay network in Fig. 1 with that of the minimum and Q-inverse equivalent relay channel receivers. The generator matrices of the 2- and 3-user relay networks are given in (49) and (50), respectively.

$$\mathbf{G_2} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{49}$$

$$\mathbf{G_3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \tag{50}$$

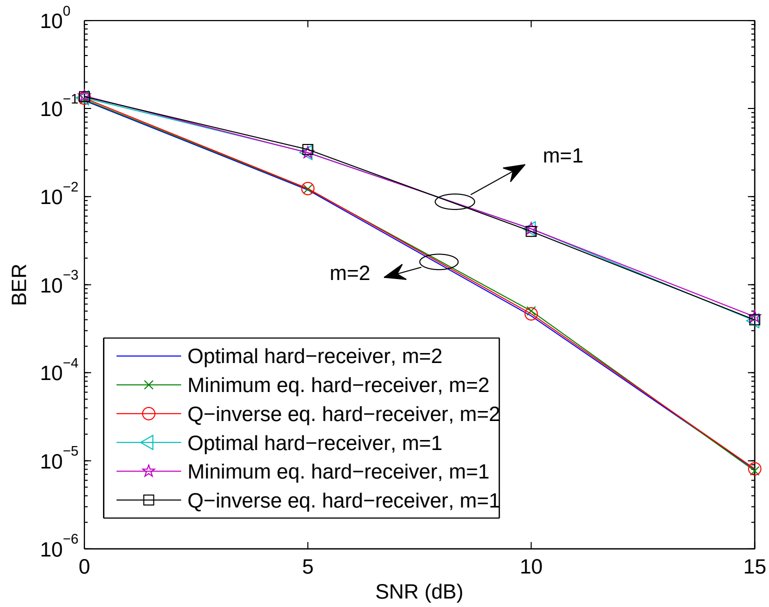

Fig. 6. 2-user GF(2) NC DetF Nakagami-m fading relay network for m=1 and m=2, hard-decision ML performance

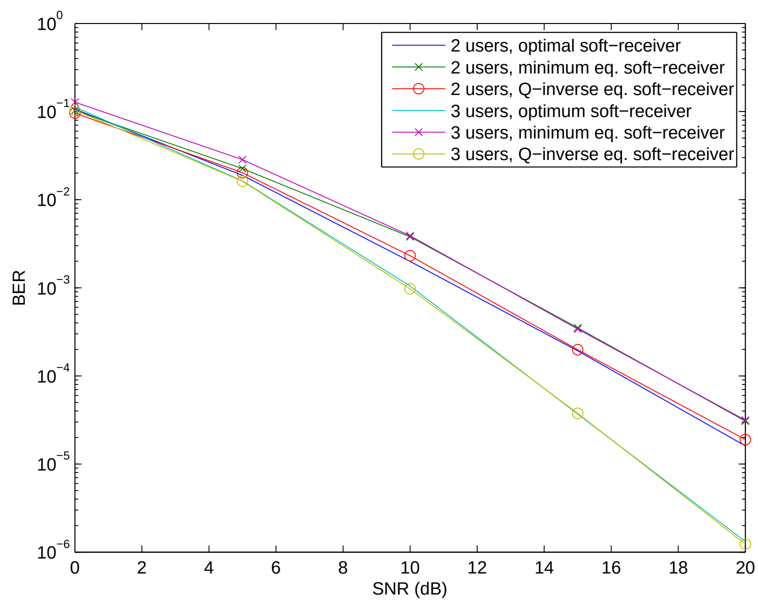

Fig. 7. 2- vs 3-user GF(2) NC DetF Nakagami-m (m=1) fading relay networks' soft-decision ML performances

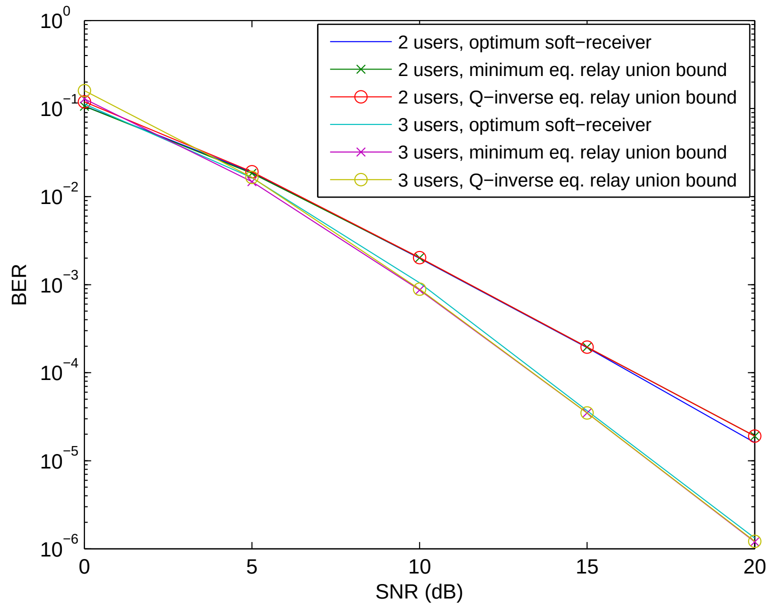

Fig. 8. 2- and 3-user GF(2) NC DetF Nakagami-m (m=1) fading relay networks' optimum soft-receiver BER performances vs minimum and Q-inverse equivalent relay networks' union bound BER performances

The Q-inverse and minimum equivalent relay channel hard-receivers are observed to display an error performance very close to each other and to the optimal hard-receiver. The diversity order is shown to increase with an increasing number of users; the diversity orders of the 2 and 3-user relay networks are 2 and 3, respectively. Fig. 6 compares the BER performance of the optimal hard-receiver of the 2-user GF(2) NC DetF Nakagami-m fading relay network in Fig. 1 with that of the minimum and Q-inverse equivalent relay channel hard-receivers for $m = 1$ and $m = 2$. The generator matrix is given in (49). It is observed that when $m = 2$, the minimum and Q-inverse equivalent channel receivers display a very close error performances to the optimal receiver and have a diversity order close to 3, compared to a diversity order of 2 when $m = 1$.

Fig. 7 compares the optimum soft-receiver BER performance of the of the 2- and 3-user GF(2) NC DetF Nakagami-m fading relay network in Fig. 1 with that of the minimum and Q-inverse soft-receivers when $m = 1$. The generator matrix used is given in (49) and (50), respectively. It is observed that the Q-inverse equivalent relay channel soft-receiver displays a very close error performance to the optimal soft-receiver, whereas the minimum equivalent relay channel soft-receiver requires 1.7 dB less SNR for a BER=10^{-5} in the 2-user case and a diversity order

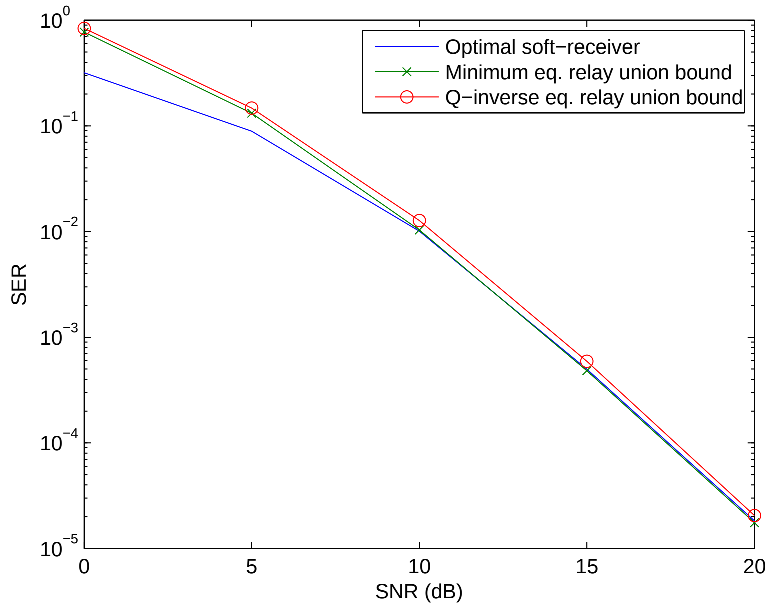

Fig. 9. 2-user GF(4) NC DetF Nakagami-m (m=1) fading relay network's optimal soft-receiver SER vs minimum and Q-inverse equivalent relay networks union bound SER performances

loss in the 3-user case.

Fig. 8 compares the analytical union bound BER performances of the 2- and 3-user GF(2) NC DetF minimum equivalent and Q-inverse equivalent relay networks with the optimum soft-receiver BER performances of the 2- and 3-user relay networks in a Nakagami-m fading environment when $m = 1$. The relay networks' generator matrices are given in (49) and (50), respectively. The minimum and Q-inverse equivalent relay networks' union bounds are very close to the relay network's optimal performance.

Fig. 9 compares the analytical union bound SER performances of the 2-user GF(4) NC DetF minimum equivalent and Q-inverse equivalent relay networks with the optimum soft-receiver's SER performance of the relay network in a Nakagami-m fading fading environment when $m = 1$. The relay network's generator matrix is given in (29). The union bound SER performances of the Q-inverse and minimum equivalent relay networks are very close to each other and approach the optimum soft-receiver SER performance at high SNR values.

Fig. 10 compares the analytical union bound BER performance of the 2-user GF(2) NC DetF Q-inverse equivalent relay network with the optimum soft-receiver's BER performance of the

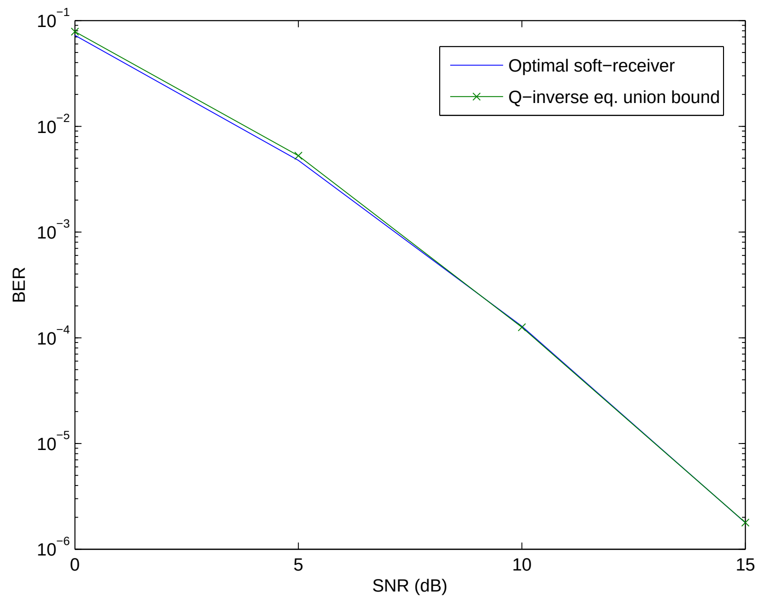

Fig. 10. 2-user GF(2) NC DetF Nakagami-m (m=2) fading relay network's optimal soft-receiver BER vs Q-inverse equivalent relay network's union bound BER performance

relay network in a Nakagami-m fading fading environment when $m = 2$. The relay network's generator matrix is given in (49). The union bound BER performance of the Q-inverse equivalent relay network is very close to the optimum soft-receiver BER performance.

V. Conclusion

This paper presents simple means of analyzing the error rate performance of a general wireless cooperative GF(q) NC DetF relay network with known relay error statistics at the destination using equivalent channel models. The approximate error rate of the relay network is derived with the use of equivalent relay channel models, where the minimum and Q-inverse equivalent relay channel models are adapted to N-user GF(q) NC DetF Nakagami-m fading relay channels. The equivalent channel models are used in developing high-performance ML receivers for Nakagami-m fading channels, which are shown to display very close error performances to the relay network's optimum receiver.

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